Pre-Semester Course Static Optimization

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2013

Organizational Issues

Course Description:
The course is intended for Ph.D. and MSQE students. The objective is to familiarize with basic concepts of unconstrained and constrained static optimization and to apply them to standard economic problems.

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Organizational Issues

Time and Place:
Location: House of Finance, Room E.20 (DZ Bank)
Time: Monday to Friday, 10:30-12:00 and 13:00-14:30

Course Website:
www.damir.stijepic.com/teaching/

Course Material:
Lectures notes and auxiliary materials are available on the course website. Problem sets will be distributed in class.

Organizational Issues

Course Outline:

Monday	10 : 15–11 : 45	Introduction
Monday	13 : 00–14 : 30	Basic Concepts
Tuesday	10 : 30–12 : 00	Existence and Uniqueness
Tuesday	13 : 00–14 : 30	Unconstrained Optimization II
Wednesday	10 : 30–12 : 00	Constrained Optimization I
Wednesday	13 : 00–14 : 30	Constrained Optimization II
Thursday	10 : 30–12 : 00	Applications I
Thursday	13 : 00–14 : 30	Applications II
Friday	10 : 30–12 : 00	Review
Friday	13 : 00–14 : 30	Questions and Discussion

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Problem Sets
Discussion of problems sets as fits. Preferably in the afternoon.

Questions and Discussion Session
Please send me your questions no later than Thursday 4pm.

Organizational Issues

Lecture notes for this course are partly based on Prof. Dr. Matthias Blonski's lecture notes for OMAT (2008).

Course material from the last year's per-semester course is available here:

http://badarinza.net/download/

Password: ****

Overview

1. Sets
   closed, open, bounded, compact, convex
2. Functions
   continuous, monotonic, convex, concave, quasi-convex

Note: This lecture is restricted to subsets of $\mathbb{R}^N$.

Set

A collection of well defined and distinct objects.

If an object e belongs to the set S, we refer to it as an element of the set S.

Notation: $e \in S$, $e \notin S$

Basics

Interior and Boundary Points

Interior Point

A point $x \in X$ is an interior point of X, if there is an $\epsilon > 0$, so that $||x - y|| < \epsilon$ implies $y \in X$.

Boundary Point

A point $x \in \mathbb{R}^N$ is a boundary point of X, if for all $\epsilon > 0$

i) $\{y | y \in \mathbb{R}^N, ||x - y|| < \epsilon\} \cap X \neq \{\}$ and
ii) $\{y | y \in \mathbb{R}^N, ||x - y|| < \epsilon\} \backslash X \neq \{\}$.

The set of all boundary points of X is said to be the boundary of the set X, denoted δX.

Basics

Open and Closed Sets

Open Set

Let $X \subset \mathbb{R}^N$. X is said to be open if for every $x \in X$ there is an $\epsilon > 0$ so that $||x - y|| < \epsilon$ implies $y \in X$.

Closed Set

Let $X \subset \mathbb{R}^N$. X is said to be closed if for every sequence x^i, $i = 1, 2, 3, ...,$ satisfying

i) $x^i \in X$ and
ii) $x^i \to x^0$,

it follows that $x^0 \in X$.

Bounded Sets

Let $X \subset \mathbb{R}^N$. X is said to be bounded if there is a $k \in \mathbb{R}$ so that
$X \subset \{x | x \in \mathbb{R}^N, |x_i| \leq k, i = 1...N\}$.

Compact Sets

Let $X \subset \mathbb{R}^N$. X is said to be compact if X is closed and bounded.

Convex Sets

Let $X \subset \mathbb{R}^N$. X is said to be *convex* if $x, y \in X$ implies $\{z | z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\} \subset X$.

Function

Let X and Y be two non-empty sets. If with each element $x \in X$ is associated one and only one element $y \in Y$, a *function f* from X to Y is defined.

We use following notations: $f : X \to Y$, $x \xrightarrow{f} y$, or $y = f(x)$.

The variable x is referred to as the *argument* of the function f, and y is referred to as the *value* of the function f at x.

The set $\{x \in X | \exists y \in Y, y = f(x)\}$ is referred as the domain, and $\{y \in Y | \exists x \in X, y = f(x)\}$ as the range of the function.

Continuous Functions

Let $f : A \to B$, $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^p$. The function f is said to be continuous at $a \in A$ if $x^\nu \in A$, $\nu = 1, 2, \ldots$ and $x^\nu \to a$ implies $f(x^\nu) \to f(a)$.

A function f is continuous on $C \subset A$, if f is continuous for all $\hat{x} \in C$.

Monotonicity

A function $f : X \to Y$ with $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}$ and $f(x_1, x_2, ...x_n) = f(x) \in \mathbb{R}$ is said to be

(i) *monotonically increasing* in x_i, if for all $x, x' \in X$ with $x'_i > x_i$ and $x'_j = x_j$ for all $j \in \{1, ..., n\} \backslash \{i\}$ $f(x') \geq f(x)$.

(ii) *monotonically decreasing* in x_i, if for all $x, x' \in X$ with $x'_i > x_i$ and $x'_j = x_j$ for all $j \in \{1, ..., n\} \backslash \{i\}$ $f(x') \leq f(x)$.

If f is monotonically increasing (decreasing) in all x_i, $i = 1, 2, ..., n$, the function f is said to be monotonically increasing (decreasing).

Replacing $\geq$ and $\leq$ with $>$ and $<$ in (i) and (ii), respectively, yields the respective definitions for strict monotonicity.

Concavity and Convexity

A function $f : X \to Y$ with the convex set $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}$ is said to be

(i) convex, if for all $x, x' \in X$ with $x \neq x'$ and all $\lambda \in (0, 1)$ $f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x')$, and

(ii) concave, if for all $x, x' \in X$ with $x \neq x'$ and all $\lambda \in (0, 1)$ $f(\lambda x + (1 - \lambda)x') \geq \lambda f(x) + (1 - \lambda)f(x')$.

Replacing $\leq$ and $\geq$ with $<$ and $<$ in (i) and (ii), respectively, yields the respective definitions for strict convexity and concavity.

Quasi-Concavity

A function $f : X \to Y$ with the convex set $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}$ is said to be quasi-concave, if for all $x, x' \in X$ with $x \neq x'$ and all $\lambda \in (0, 1)$
$f(\lambda x + (1 - \lambda)x') \geq \min\{f(x), f(x')\}$.

Replacing $\geq$ with $>$, yields the definition for strict quasi-concavity.

Overview

1. Local and Global Extrema
2. Existence and Uniqueness

General Remarks

Local and Global Extrema

Local and Global Extrema

Let $f : \mathbb{R}^N \to \mathbb{R}$. The element $\hat{x} \in A \subset Def(f)$ is said to be a local maximum (minimum) on A if and only if $\exists \epsilon > 0 : \forall x \in \{x \mid ||x - \hat{x}|| \leq \epsilon\} \wedge A$ it holds: $f(x) \leq (\geq) f(\hat{x})$.

The element $\hat{x} \in A \subset Def(f)$ is said to be a global maximum (minimum) on A if and only if $\forall x \in A$ it holds: $f(x) \leq (\geq) f(\hat{x})$.

General Remarks

Existence and Uniqueness

Existence (Weierstrass)

Let f be a function from S to T. If f is continuous on S, and if S is compact, non-empty, then $f(S)$ has a maximum and a minimum.

General Remarks

Existence and Uniqueness

Uniqueness

Let $A \subset Def(f) \subset \mathbb{R}^N$ be a convex set and let $f : \mathbb{R}^N \to \mathbb{R}$ be strictly quasi-concave on A. Then there exists at most one solution to the optimization problem $\max_{x \in A} f(x)$.

Existence and Uniqueness

Let $A \subset Def(f) \subset \mathbb{R}^N$ be a non-empty, compact and convex set and let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous and strictly quasi-concave on A. Then there exists exactly one solution to the optimization problem $\max_{x \in A} f(x)$.

Unconstrained Optimization

Overview

Overview

1. Necessary and Sufficient Conditions for Local Extrema
2. Sufficient Conditions for Global Extrema

Unconstrained Optimization

Necessary and Sufficient Conditions

Local Extrema

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a C^2-function with $Def(f) = \mathbb{R}^N$. It holds

1. If $\hat{x}$ is a local maximum, then $\nabla f(\hat{x}) = 0$ and $Hf(\hat{x})$ is negative semidefinite.
2. If $\hat{x}$ is a local minimum, then $\nabla f(\hat{x}) = 0$ and $Hf(\hat{x})$ is positive semidefinite.
3. If $\nabla f(\hat{x}) = 0$ and $Hf(\hat{x})$ is negative definite, then $\hat{x}$ is a local maximum.
4. If $\nabla f(\hat{x}) = 0$ and $Hf(\hat{x})$ is positive definite, then $\hat{x}$ is a local minimum.

Unconstrained Optimization

Sufficient Conditions for Global Extrema

Global Extrema

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a C^2-function and let the function be concave. It holds: $\nabla f(\hat{x}) = 0 \Leftrightarrow \hat{x}$ is a global maximum.

Overview

1. Lagrange Function and Kuhn-Tucker Conditions
2. Necessary and Sufficient Conditions
3. Equality Constraints

Constrained Optimization

Optimization Problem

Optimization Problem

$$\max_{x} f(x) \text{ subject to } g_j(x) \geq 0 \text{ for } j = 1, ..., m, \quad (1)$$

where the functions $f, g_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be differentiable.

Constrained Optimization

Lagrange Function

Lagrange Function

The function

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \qquad (2)$$

is said to be the the Lagrange function of the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) \geq 0 \text{ for } j = 1, ..., m. \qquad (3)$$

The variable λ_j is said to be the Lagrange multiplier of the constraint $g_j(x) \geq 0$.

Kuhn-Tucker Conditions

The conditions

1. $\nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0$,
2. $g_j(x) \geq 0$ for $j = 1, ..., m$,
3. $\lambda_j \geq 0$ for $j = 1, ..., m$, and
4. $\lambda_j g_j(x) = 0$ for $j = 1, ..., m$,

are said to be the Kuhn-Tucker conditions (KT) of the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) \geq 0 \text{ for } j = 1, ..., m. \tag{4}$$

Constrained Optimization

Constraint Qualification

Constraint Qualification

The point $\hat{x}$ satisfies the Constraint Qualification (CQ) if and only if the gradient vectors $\nabla g_j(\hat{x})$ are linearly independent for the $j \in \{1, ..., m\}$, where the respective constraints are binding, i.e. $g_j(\hat{x}) = 0$.

Constrained Optimization

Necessary Conditions

Necessary Conditions

Let $f, g_j : \mathbb{R}^N \to \mathbb{R}$ be differentiable functions at the relevant points. Let $\hat{x}$ be a solution of the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) \geq 0 \text{ for } j = 1, ..., m, \tag{5}$$

and let $\hat{x}$ satisfy the Constraint Qualification. Then there are Lagrange multipliers $\hat{\lambda}$, so that for $(\hat{x}, \hat{\lambda})$ the Kuhn-Tucker conditions are satisfied.

Constrained Optimization

Sufficient Conditions

Sufficient Conditions

Let $f, g_j : \mathbb{R}^N \rightarrow \mathbb{R}$ be at the relevant points differentiable functions, and let f be concave and g_j quasi-concave. Furthermore, let $(\hat{x}, \hat{\lambda})$ satisfy the Kuhn-Tucker conditions. Then $\hat{x}$ is a solution to the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) \geq 0 \text{ for } j = 1, ..., m. \tag{6}$$

Constrained Optimization

Necessary Conditions

Necessary Conditions

Let $f, g_j : \mathbb{R}^N \to \mathbb{R}$ be differentiable functions at the relevant points. Let $\hat{x}$ be a solution of the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) = 0 \text{ for } j = 1, ..., m, \tag{7}$$

and let $\hat{x}$ satisfy the Constraint Qualification. Then there are Lagrange multipliers $\hat{\lambda}$, so that for $(\hat{x}, \hat{\lambda})$ the simplified Kuhn-Tucker (KT') conditions

1. $\nabla f(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) = 0$, and
2. $g_j(x) = 0$ for $j = 1, ..., m$,

are satisfied.

Constrained Optimization

Sufficient Conditions

Sufficient Conditions

Let $f, g_j : \mathbb{R}^N \to \mathbb{R}$ be at the relevant points differentiable functions, and let f be concave and g_j linear functions. Furthermore, let $(\hat{x}, \hat{\lambda})$ satisfy the simplified Kuhn-Tucker conditions (KT'). Then $\hat{x}$ is a solution to the maximization problem

$$\max_x f(x) \text{ subject to } g_j(x) = 0 \text{ for } j = 1, ..., m. \tag{8}$$